Generalised triangle groups of type $(3, 3, 2)$

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Abstract

If G is a group with a presentation of the form $\langle x, y | x^3 = y^3 = W(x, y)^2 = 1\rangle$, then either G is virtually soluble or G contains a free subgroup of rank 2. This provides additional evidence in favour of a conjecture of Rosenberger.

1 Introduction

A *generalised triangle group* is a group G with a presentation of the form

$$\langle x, y | x^p = y^q = W(x, y)^r = 1\rangle$$

where $p, q, r \geq 2$ are integers and $W(x, y)$ is a word of the form $x^{\alpha(1)}y^{(1)} \cdots x^{\alpha(k)}y^{(k)}$, $(0 < \alpha(i) < p$, $0 < (i) < q)$. We say that G is of *type* (p, q, r). The parameter k is called the *length*. Without loss of generality, we assume that $p \leq q$.

A conjecture of Rosenberger [19] asserts that a Tits alternative holds for generalised triangle groups:

Conjecture A (Rosenberger) *Let G be a generalised triangle group. Then either G is soluble-by-finite or G contains a non-abelian free subgroup.*

This conjecture has been verified in a large number of special cases. (See for example the survey in [9].) In particular it is now known:

- when $r \geq 3$ [8];
- when $\frac{1}{p} + \frac{1}{q} \geq \frac{1}{2}$ [3, 13];
- when $q \geq 6$ [18, 21, 4, 5, 1, 7, 15];
- when $k \leq 6$ [19, 18, 20];
- for $(p, q, r) = (3, 4, 2)$ [2, 16];
- for $(p, q, r) = (2, 4, 2)$ and k odd [6].

In the present note we describe a proof of the Rosenberger Conjecture for the case $(p,q,r) = (3,3,2)$. Using essentially the same argument, we also prove the Conjecture in the case where $(p,q,r) = (2,3,2)$ and k is even – with the exception of 6 groups that our methods are unable to handle.

The proofs rely to some extent on computations using the computer algebra package GAP [10]. The strategy of proof, however, is straightforward. Firstly, a theoretical analysis shows that, if G is a generalised triangle group G of type $(3,3,2)$ that does not contain a non-abelian free subgroup, then the corresponding *trace polynomial* must have a very restricted form. In particular this analysis provides a bound $k \leq 20$ for the length parameter k of such a group. Secondly, a computer search finds all words of length up to 20 whose trace polynomial has this restricted form. There turn out to be precisely 19 such words (up to a standard equivalence relation), of which 8 have length $k \leq 6$: the conjecture is already known for the 8 groups corresponding to these short words. Finally, it is observed that, in the remaining 11 cases, a small cancellation condition applies to G (regarded as a quotient of $\mathbb{Z}_3 * \mathbb{Z}_3$). We complete the proof by showing that the small cancellation condition implies the existence of a non-abelian free subgroup. The small cancellation arguments applied to do this yield somewhat more general results which may be of independent interest, so we present these in a more general form in § 2 below.

The theoretical analysis of the case $(p,q,r) = (2,3,2)$, k even, is identical, subject to two provisos. Firstly, an equivalence class of words in the $(3,3,2)$ case can correspond to either one or two equivalence classes of words of even length in the $(2,3,2)$ case. (Here *equivalence* refers to some standard moves on words W that do not change the trace polynomial or the isomorphism type of the resulting group. See §3 for details.) Secondly, these words are twice as long as their $(3,3,2)$ counterparts. The latter difference means that fewer of them are already dealt with by existing results.

Section 2 below contains the small-cancellation results mentioned above. Sections 3 and 4 contain respectively a discussion of the equivalence relation on words, and some elementary results on their trace polynomials. The proof of the main result on generalised triangle groups of type $(3,3,2)$ is in Section 5, and in Section 6 we discuss the variations needed to attack generalised triangle groups of type $(2,3,2)$ with even length parameter. Section 7 contains some remarks about the computational aspects of the work, including a description of the search algorithm. Logs of GAP sessions performing some of the calculations are contained in an Appendix. Tables at the end of the paper list all words (up to equivalence) whose trace polynomials do not immediately imply the existence of free subgroups in the corresponding group. An ancillary file attached to this preprint contains the GAP code listings used in the search algorithm.

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2 Small Cancellation

In this section we prove two results on one-relator products of groups where the relator satisfies a certain small cancellation condition. We will apply these specifically to generalised triangle groups of types $(3,3,2)$ and $(2,3,2)$ respectively, but as the results seem of independent interest, we prove them in the widest generality available.

Suppose that Γ_1, Γ_2 are groups, and $U \in \Gamma_1 * \Gamma_2$ is a cyclically reduced word of length at least 2. (Here and throughout this section, *length* means length in the free product sense.) A word

$V \in \Gamma_1 * \Gamma_2$ is called a *piece* if there are words V', V'' with $V' \neq V''$, such that each of $V \cdot V'$, $V \cdot V''$ is cyclically reduced as written, and each is equal to a cyclic conjugate of U or of U^{-1}. A cyclic subword of U is a *non-piece* if it is not a piece.

By a *one-relator product* $(\Gamma_1 * \Gamma_2)/U$ of groups Γ_1, Γ_2 we mean the quotient of their free product $\Gamma_1 * \Gamma_2$ by the normal closure of a cyclically reduced word U of positive length. Recall [12] that a *picture* over the one-relator product $G = (\Gamma_1 * \Gamma_2)/U$ is a graph $\mathcal{P}$ on a surface Σ (which for our purposes will always be a disc) whose corners are labelled by elements of $\Gamma_1 \cup \Gamma_2$, such that

1. the label around any vertex, read in clockwise order, spells out a cyclic permutation of U or U^{-1};

2. the labels in any region of $\Sigma \setminus \mathcal{P}$ either all belong to Γ_1 or all belong to Γ_2;

3. if a region has k boundary components labelled by words $W_1, \dots, W_k \in \Gamma_i$ (read in anti-clockwise order; with $i = 1, 2$), then the quadratic equation

$$\prod_{j=1}^{k} X_j W_j X_j^{-1} = 1$$

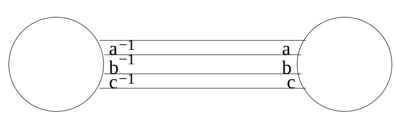
   is solvable for $X_1, \dots, X_k$ in Γ_i. (In particular, if $k = 1$ then $W_1 = 1$ in Γ_i).

Note that edges of $\mathcal{P}$ may join vertices to vertices, or vertices to the boundary $\partial\Sigma$, or $\partial\Sigma$ to itself, or may be simple closed curves disjoint from the rest of $\mathcal{P}$ and from $\partial\Sigma$.

The *boundary label* of $\mathcal{P}$ is the product of the labels around $\partial\Sigma$. By a version of van Kampen's Lemma, there is a picture with boundary label $W \in \Gamma_1 * \Gamma_2$ if and only if W belongs to the normal closure of U.

A picture is *minimal* if it has the fewest possible vertices among all pictures with the same (or conjugate) boundary labels. In particular every minimal picture is *reduced*: no edge e joins two distinct vertices in such a way that the labels of these two vertices that start and finish at the endpoints of e are mutually inverse.

In a reduced picture, any collection of parallel edges between two vertices (or from one vertex to itself) corresponds to a collection of consecutive 2-gonal regions, and the labels within these 2-gonal regions spell out a piece:



Since U is cyclically reduced, no corner of an interior vertex is contained in a 1-gonal region.

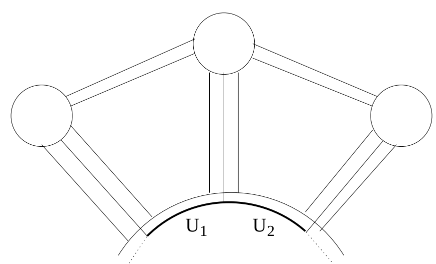
Theorem 2.1 *Let ℓ be an even positive integer. Suppose that $U \equiv U_1 \cdot U_2 \cdot U_3 \cdot U_4 \cdot U_5 \cdot U_6 \in \Gamma_1 * \Gamma_2$ with each U_i a non-piece of length at least ℓ. Suppose also that $A, B \in \Gamma_1 * \Gamma_2$ are reduced words of length ℓ such that A is not equal to any cyclic conjugate of $B^{\pm 1}$ and such that no U_i is equal to a subword of a power of A. Then $G := (\Gamma_1 * \Gamma_2)/\langle\langle U \rangle\rangle$ contains a non-abelian free subgroup.*

Proof. Since ℓ is even and positive, any reduced word of length ℓ in $\Gamma_1 * \Gamma_2$ is cyclically reduced. Replacing A by A^{-1} and/or B by B^{-1} if necessary, we may assume that each of A, B begins with a letter from Γ_1 and ends with a letter from Γ_2. Choose a large positive integer $N > 20K\ell$, where

K is the length of U, and define $X := A^N B^N$, $Y := B^N A^N$. We claim that X, Y freely generate a free subgroup of G.

We prove this claim by contradiction. Suppose that $Z(X,Y)$ is a non-trivial reduced word in X, Y such that $Z(X,Y) = 1$ in G. Then there exists a picture $\mathcal{P}$ on the disc D^2 over the one-relator product G with boundary label $Z(X,Y)$. Without loss of generality, we may assume that $\mathcal{P}$ is minimal, hence reduced.

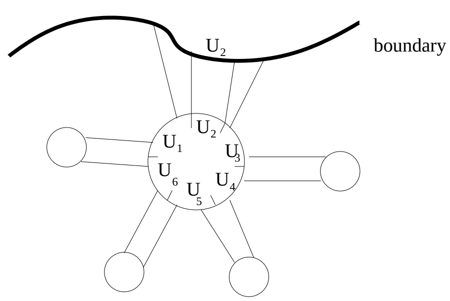
Suppose that v is an interior vertex of $\mathcal{P}$. The vertex label of v is U or U^{-1} – by symmetry we can assume it is U. The subword U_1 of U corresponds to a sequence of consecutive corners of v; at least one of these corners does not belong to a 2-gonal region of $\mathcal{P}$, since U_1 is a non-piece. It follows that at least one of the corners of v within the subword U_1 of the vertex label does not belong to a 2-gonal region. The same follows for the subwords $U_2, \ldots, U_6$, so v has at least 6 non-2-gonal corners.

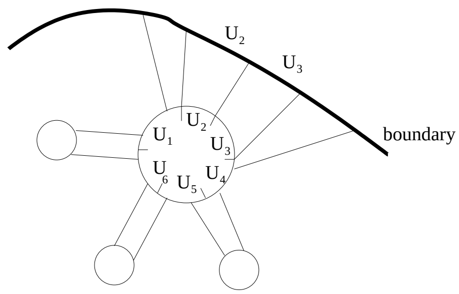


Now consider the (cyclic) sequence of *boundary* (that is, non-interior) vertices of $\mathcal{P}$, $v_1, \ldots, v_n$ say. This is intended to mean that the closed path ∂D^2, with an appropriate choice of starting point, meets a sequence of arcs that go to v_1, separated by 2-gons, then a sequence of arcs that go to v_2, separated by 2-gons, and so on, finishing with a sequence of arcs that go to v_n, separated by 2-gons, before returning to its starting point. Note that it is possible that an arc of $\mathcal{P}$ joins two points on ∂D^2; any such arc is ignored here. Note also that we do not insist that $v_i \neq v_j$ for $i \neq j$ in general. It is possible for the sequence of boundary vertices to visit a vertex v several times. Nevertheless it is important to regard such visits as pairwise distinct, so the notation $v_1, v_2, \ldots$ is convenient. We say that a boundary vertex is *simple* if it appears only once in this sequence.

If v_j is connected to ∂D^2 by k arcs separated by $k-1$ 2-gons, then this corresponds to a common (cyclic) subword W_j of $Z(X,Y)$ and U, of length $k-1$. Let $\kappa(j) \leq 6$ be the maximum integer t such that, for some $s \in \{1, \ldots, 6\}$, W_j contains a subword equal to $(U_s \cdot U_{s+1} \cdots U_{s+t})^{\pm 1}$ (indices modulo 6). If no such t exsits, we define $\kappa(j) = -1$.

If v_j is a simple boundary vertex with only $r \leq 4$ corners not belonging to 2-gons, then it is easy to see that $\kappa(j) \geq 5 - r$:





There are more complex rules for non-simple boundary vertices. Nevertheless, it is an easy consequence of Euler's formula, together with the fact that interior vertices have 6 or more non-2-gonal corners, that

$$\sum_{j=1}^{n} \kappa(j) \geq 6.$$

Now consider the word $Z(X,Y)$ as a cyclic word in $\Gamma_1 * \Gamma_2$. Where a letter $X = A^N B^N$ or $Y = B^N A^N$ is followed by another letter X or Y, then there is no cancellation in $\Gamma_1 * \Gamma_2$. Similarly there is no cancellation where X^{-1} or Y^{-1} is followed by X^{-1} or Y^{-1}. Where X is followed by Y^{-1} or *vice versa*, or where Y is followed by X^{-1} or *vice versa*, then there is possible cancellation, but since $A \neq B$ the amount of cancellation is limited to at most ℓ letters from either side.

If Z has length L as a word in $\{X^{\pm 1}, Y^{\pm 1}\}$, then after cyclic reduction in $\Gamma_1 * \Gamma_2$ it consists of L subwords of the form $A^{\pm(N-1)}$, L subwords of the form $B^{\pm(N-1)}$, and L subwords $V_1, \ldots, V_L$, each of length at most 2ℓ.

Now suppose that v_j is a boundary vertex of $\mathcal{P}$ with $\kappa(j) \geq 0$. Then $U_i^{\pm 1}$ is equal to a subword of W_j for some i. Since U_i cannot be a subword of a power of A, W_j is not entirely contained within one of the segments labelled $A^{\pm(N-1)}$.

If, in addition, $\kappa(j) > 0$, then W_j has a subword of of the form $(U_i U_{i+1})^{\pm 1}$ (subscripts modulo 6) As above, W_j cannot be contained in one of the subwords $A^{\pm(N-1)}$. If it is contained in a subword of $B^{\pm(N-1)}$, then it is a periodic word of period ℓ (that is, its i-th letter is equal to its $(i+\ell)$-th letter for all i for which this makes sense). Since U_{i+1} has length at least ℓ, there are at least two distinct subwords of $U_i U_{i+1}$ equal to U_i, contradicting the fact that U_i is a non-piece in U.

Thus we see that the subwords W_j of $Z(X,Y)$ corresponding to boundary vertices v_j with $\kappa(j) > 0$ can occur only at certain points of $Z(X,Y)$: where an $A^{\pm(N-1)}$-segment meets a $B^{\pm(N-1)}$-segment; or at part of one of the words V_i.

In particular, the number of boundary vertices v_j with $\kappa(j) > 0$ is bounded above by $L(2\ell+1)$. It follows that

$$\kappa := \sum_{j} \kappa(j) \leq 5L(2\ell + 1),$$

where the sum is taken over those boundary vertices v_j with $\kappa(j) \geq 0$.

The goal is to show that the total positive contribution to the sum κ from those v_j with $\kappa(j) > 0$ is cancelled out by negative contributions to κ from other boundary vertices. This will show that $\kappa \leq 0$, contradicting the assertion above that $\kappa \geq 6$.

Recall that K is the length of U. Thus each $A^{\pm(N-1)}$-segment of $\partial\mathcal{P}$ is joined to at least $(N-1)\ell/K$ boundary vertices, at most 2 of which (those at the ends of the segment) can make non-negative contributions to κ. The remaining vertices each contribute at most -1 to κ. Since

$N > 20K\ell$, it follows that the negative contributions outweigh the positive contributions, as required.

This gives the desired contradiction, which proves the theorem.

Corollary 2.2 *Let Γ_1 and Γ_2 be groups, and suppose $x \in \Gamma_1$ and $y \in \Gamma_2$ are elements of order greater than 2. Suppose that $W \equiv U_1 \cdot U_2 \cdot U_3 \in \Gamma_1 * \Gamma_2$ with each U_i a non-piece of length at least 4. Then $G = (\Gamma_1 * \Gamma_2)/\langle\langle W^2 \rangle\rangle$ contains a non-abelian free subgroup.*

Proof. Let $A_1 = xyxy$, $A_2 = xy^{-1}xy^{-1}$, $A_3 = xyxy^{-1}$ and $A_4 = xyx^{-1}y^{-1}$. Then for $i \neq j$, A_i is not equal to a cyclic conjugate of $A_j^{\pm 1}$. Hence if (say) U_1 is equal to a subword of a power of A_i, it cannot be equal to a subword of a power of A_j. Hence there is at least one $A \in \{A_i,\ 1 \leq i \leq 4\}$ with the property that no U_i is equal to a subword of a power of A. Now choose $B \in \{A_i,\ 1 \leq i \leq 4\} \setminus A$ and apply the theorem, with $U_4 = U_1$, $U_5 = U_2$ and $U_6 = U_3$.

Corollary 2.3 *Let Γ_1 and Γ_2 be groups, and suppose $x \in \Gamma_1$ has order 2 and $y \in \Gamma_2$ has order greater than 2. Suppose that $W \equiv U_1 \cdot U_2 \cdot U_3 \in \Gamma_1 * \Gamma_2$ with each U_i a non-piece of length at least 8. Then $G = (\Gamma_1 * \Gamma_2)/\langle\langle W^2 \rangle\rangle$ contains a non-abelian free subgroup.*

Proof. Let $A_1 = xyxyxyxy$, $A_2 = xyxy^{-1}xyxy^{-1}$, $A_3 = xyxyxyxy^{-1}$ and $A_4 = xyxyxy^{-1}xy^{-1}$. As in the previous proof, we can choose $A, B \in \{A_1, A_2, A_3, A_4\}$ such that no U_i is equal to a subword of a power of A, and A is not equal to a cyclic conjugate of $B^{\pm 1}$, and apply the theorem, with $U_4 = U_1$, $U_5 = U_2$ and $U_6 = U_3$.

3 Equivalence of words

Our object of study is a group

$$G = \langle x, y | x^p = y^q = W(x,y)^r = 1 \rangle$$

where

$$W(x,y) = x^{\alpha(1)} y^{(1)} \cdots x^{\alpha(k)} y^{(k)},$$

and $0 < \alpha(i) < p$, $0 < (i) < q$ for each i.

We think of the word W as a cyclically reduced word in the free product

$$\mathbb{Z}_p * \mathbb{Z}_q = \langle x, y | x^p = y^q = 1 \rangle.$$

We regard two such words W, W' as *equivalent* if one can be transformed to the other by moves of the following types:

- cyclic permutation of W,
- inversion of W,
- automorphism of $\mathbb{Z}_p$ or of $\mathbb{Z}_q$, and
- (if $p = q$) interchange of x, y.

It is clear that, if W, W' are equivalent words, then the resulting groups

$$G = \langle x, y | x^p = y^q = W(x,y)^r = 1\rangle$$

and

$$G' = \langle x, y | x^p = y^q = W'(x,y)^r = 1\rangle$$

are isomorphic. Hence for the purposes of studying the Rosenberger Conjecture (Conjecture A) it is enough to consider words up to equivalence.

4 Trace Polynomials

Suppose that $X, Y \in SL(2,\mathbb{C})$ are matrices, and $W = W(X,Y)$ is a word in X, Y. Then the trace of W can be calculated as the value of a 3-variable polynomial, where the variables are the traces of X, Y and XY [11]. We can use this to find and analyse *essential representations* from G to $PSL(2,\mathbb{C})$. (A representation of G is *essential* if the images of $x, y, W(x,y)$ have orders p, q, r respectively.)

We can force the images x, y to have orders p, q in $PSL(2,\mathbb{C})$ by mapping them to matrices $X, Y \in SL(2,\mathbb{C})$ of trace $2\cos(\pi/p)$ and $2\cos(\pi/q)$ respectively. Then the trace of $W(X,Y) \in SL(2,\mathbb{C})$ is given by a one-variable polynomial $\tau_W(l)$, where l denotes the trace of XY. Since we are in practice interested in the case where $r = 2$, we obtain an essential representation by choosing l to be a root of τ_W.

We recall here some properties of τ_W. Details can be found, for example, in [9].

- τ_W has degree k;
- when $p, q \leq 3$, $\tau_W(l)$ is monic and has integer coefficients;
- when $p = 2$, τ_W is an odd or even polynomial, depending on the parity of k.

Lemma 4.1 *If $p = 2$, $q = 3$ and W, W' are equivalent, then $\tau_W(l) = \tau_{W'}(l)$.*
If $p = q = 3$ and W, W' are equivalent of length k, then either $\tau_W(l) = \tau_{W'}(l)$ or $\tau_W(l) = (-1)^k \tau_{W'}(1-l)$.

Proof. Since the trace of a matrix is a conjugacy invariant, it follows that the trace polynomial is unchanged by cyclically permuting W. Moreover, if $X \in SL(2,\mathbb{C})$ then the traces of X, X^{-1} are equal, so the trace polynomial is unchanged by inverting W.

Suppose first that $p = 2$ and $q = 3$. Then we cannot interchange x and y. Moreover, there is no nontrivial automorphism of $\mathbb{Z}_2$ and only one nontrivial automorphism of $\mathbb{Z}_3$, which replaces y by y^2. If $tr(X) = 0$ and $tr(Y) = 1$, then $tr(Y^{-1}) = 1$, and

$$tr(XY^{-1}) + tr(XY) = tr(X)tr(Y) = 0,$$

so $tr(XY^{-1}) = -tr(XY) = -l$, so

$$tr(W(X,Y^2)) = tr(W(X,-Y^{-1})) = (-1)^k \tau_W(-l) = \tau_W(l).$$

In other words, this does not change $\tau_W(l)$, as claimed.

Now suppose that $p = q = 3$. If $tr(X) = 1 = tr(Y)$, then $tr(Y^{-1}) = 1$ also. Interchanging x, y in W has the effect on $\tau_W(l) = tr(W(X,Y))$ of replacing $l = tr(XY)$ by $tr(YX) = l$ – in other words, no change.

Now in this case

$$tr(XY^{-1}) + tr(XY) = tr(X)tr(Y) = 1.$$

Hence replacing y by y^2 has the effect of replacing $\tau_W(l) = tr(W(X,Y)$ by

$$tr(W(X,Y^2)) = tr(W(X,-Y^{-1})) = (-1)^k tr(W(X,Y^{-1})) = (-1)^k \tau_W(1-l),$$

as claimed.

Lemma 4.2 *Let W be a cyclically reduced word in $\mathbb{Z}_3 * \mathbb{Z}_3 = \langle x, y | x^3 = y^3 = 1\rangle$, and define $Z(u,v) = W(uvu, v) \in \mathbb{Z}_2 * \mathbb{Z}_3 = \langle u, v | u^2 = v^3 = 1\rangle$. Then $\tau_Z(\lambda) = (-1)^k \tau_W(2 - l^2)$.*

Proof. Let U, V be matrices with $tr(U) = 0$, $tr(V) = 1$. Define $X = V$ and $Y = -UVU$ so that $tr(X) = 1 = tr(Y)$, and $tr(XY) = -tr((UV)^2) = 2 - l^2$ where $l = tr(UV)$. Hence

$$\tau_Z(l) = tr(Z(U,V)) = tr(W(UVU,V)) = (-1)^k tr(W(X,Y)) = (-1)^k \tau_W(2 - l^2)$$

as claimed

Theorem 4.3 *Let $G = \langle x, y | x^3 = y^3 = W(x,y)^2 = 1\rangle$ where $W = x^{\alpha(1)}y^{(1)} \cdots x^{\alpha(k)}y^{(k)}$ with $\alpha(i), (i) \in \{1,2\}$ for each i. If G does not contain a free subgroup of rank 2, then $\tau_W(\lambda)$ has the form*

$$\tau_W(\lambda) = \lambda^a(\lambda-1)^b(\lambda^2-\lambda-1)^c$$

with $a, b \le 1$ and $c \le 3(a+b+1)$. In particular $k = a + b + 2c \le 20$.

Proof. If λ_0 is a root of the trace polynomial, then there is an essential representation $\rho : G \to PSL(2,\mathbb{C})$ such that $\rho(x), \rho(y)$ are represented by matrices of trace 1 and $\rho(xy)$ is represented by a matrix of trace λ_0. If the image $\rho(G)$ of ρ is non-elementary, then $\rho(G)$, and hence also G, contains a free subgroup of rank 2, contrary to hypothesis.

Hence every essential representation $G \to PSL(2,\mathbb{C})$ has elementary image. But the only elementary subgroups of $PSL(2,\mathbb{C})$ generated by two elements of order 3 that contain elements of order 2 are isomorphic to A_4 (corresponding to roots 0 or 1 of τ_W) and A_5 (corresponding to roots $\frac{1\pm\sqrt{5}}{2}$).

Since τ_W has integer coefficients, the two potential roots $\frac{1\pm\sqrt{5}}{2}$ occur with equal multiplicities. Since $p = q = 3$, τ_W is monic. Thus τ_W has the form

$$\tau_W(\lambda) = \lambda^a(\lambda-1)^b(\lambda^2-\lambda-1)^c$$

for some non-negative integers a, b, c.

To obtain the desired bounds on a, b, c we use the following observation. The space $\mathcal{M}_1$ of matrices in $SU(2) \subset SL_2(\mathbb{C})$ with trace 1 is path-connected. (Indeed, it is homeomorphic to the 2-sphere S^2.) For any $X \in \mathcal{M}_1$, we can vary Y continuously in $\mathcal{M}_1$ from X to X^{-1}, and $\lambda = tr(XY)$ will vary continuously from $-1 = tr(XX)$ to $2 = tr(XX^{-1})$. By the Intermediate Value Theorem, any $\lambda \in [-1,2]$ can be realised as $tr(XY)$ for some choice of $X, Y \in \mathcal{M}_1$. But

for $X, Y \in \mathcal{M}_1$ we have $W(X,Y) \in SU(2)$, so $\tau_W(l) = tr(W(X,Y)) \in [-2,2]$. This shows that $|\tau_W(l)| \le 2$ for $-1 \le l \le 2$. Now $|\tau_W(2)| = 2^a$ and $|\tau_W(-1)| = 2^b$, so $a \le 1$ and $b \le 1$. Finally,

$$\left|\tau_W\left(\frac{1}{2}\right)\right| = \left(\frac{5}{4}\right)^c \left(\frac{1}{2}\right)^{a+b}.$$

From this we deduce that

$$c \ln(5) \le (a+b+2c+1)\ln(2),$$

which implies the desired conclusion

$$c \le 3(a+b+1)$$

given that $a+b \in \{0,1,2\}$.

Essentially the same proof gives the following parallel version:

Theorem 4.4 *Let $G = \langle u, v | u^2 = v^3 = W(u,v)^2 = 1\rangle$ where $W = uv^{\alpha(1)} \cdots uv^{\alpha(k)}$ with $\alpha(i) \in \{1,2\}$ for each i and k even. If G does not contain a free subgroup of rank 2, then $\tau_W(\lambda)$ has the form*

$$\tau_W(\lambda) = (\lambda^2-1)^a(\lambda^2-2)^b(\lambda^4-3\lambda^2+1)^c$$

with $a, b \le 1$ and $c \le 3(a+b+1)$. In particular $k = 2a+2b+4c \le 40$.

5 The main result

Theorem 5.1 *Let $G = \langle x, y | x^3 = y^3 = W(x,y)^2 = 1\rangle$ be a generalised triangle group of type $(3,3,2)$. Then the Rosenberger Conjecture holds for G: either G is soluble-by-finite, or G contains a non-abelian free subgroup.*

Proof. Write

$$W = x^{\alpha(1)} y_{\_}^{(1)} \cdots x^{\alpha(k)} y_{\_}^{(k)}.$$

A computer search produces a list of all words W, up to equivalence, for which the trace polynomial τ_W has the form indicated in Theorem 4.3: see Table 1. If W is not equivalent to a word in the list, then G has a nonabelian free subgroup by Theorem 4.3, so we may restrict our attention to the words W in Table 1.

For those W in Table 1 for which $k \ge 7$ (namely, numbers 9-19) the small cancellation hypotheses of Corollary 2.2 are satisfied, and so G contains a nonabelian free subgroup.

For $k \le 6$ (words 1-8) in the table, the result is known. Specifically, groups 1-3 are well-known to be finite of orders 12, 180 and 288 respectively; groups 4-6 were proved to have nonabelian free subgroups in [18]; and finally groups 7 and 8 were shown in [20] to be *large*. (That is, each contains a subgroup of finite index which admits an epimorphism onto a non-abelian free group.) Since [20] is an unpublished thesis, we will give, for each result we cite from [20], either a GAP calculation reproducing Williams' argument, or an independent proof. In particular, a GAP calculation following the proof in [20] for Group 7 is shown in the Appendix. The largeness of Group 8 in Table 1 follows from the largness of the corresponding group in Table 2, as discussed in § 6 below. The latter group is shown to be large using a separate GAP calculation, which is also reproduced in the Appendix.

This completes the proof.

6 Variation: type $(2,3,2)$ with even length

The group $G = \langle x, y | x^3 = y^3 = W(x,y)^2 = 1\rangle$ has a homomorphic image $\overline{G}$ which is an index-2 subgroup of $H = \langle u, v | u^2 = v^3 = W(uvu, v)^2 = 1\rangle$. Clearly H is a generalised triangle group of type $(2,3,2)$ whose length parameter k is twice that of G. Conversely, every generalised triangle group of type $(2,3,2)$ with even length parameter arises in this way. There is thus at least a superficial parallel between generalised triangle groups of type $(2,3,2)$ with even length parameter and those of type $(3,3,2)$. The two types can be analysed in entirely analogous ways. In particular, the same computer search used to list the possible relators in G yields also the possible relators in H (see Table 2). However, we must take care over a few details.

1. Interchanging y, y^2 in $W(x,y)$ produces a word $W'(x,y)$ that is equivalent to $W(x,y)$. However, $W'(uvu,v) = W(uvu,v^2)$ is not in general equivalent to $W(uvu,v)$. Thus each candidate for $W(x,y)$ in G gives rise to either one or two candidates for $W(uvu,v)$ in H (up to equivalence). This is reflected in the numeration of words in Table 2: for example, word 1, $W(x,y) = xy$ in Table 1 gives rise to words 1a, $W(uvu,v) = uvuv$, and 1b, $W(uvu,v^2) = uvuv^2$ in Table 2. (Where the two words $W(uvu,v)$ and $W(uvu,v^2)$ are equivalent, only one is shown in Table 2.)

2. If H contains a nonabelian free subgroup, then so does $\overline{G}$, and hence so does G. We have already used this explicitly in the proof of Theorem 5.1: taking H to be Group 8 in Table 2, we show in the Appendix that H is large. In this case G is Group 8 of Table 1, which we also deduced to be large. The converse implication does not necessarily hold, however. This is most graphically illustrated by the case of Group 4 in Table 1. As mentioned in the proof of Theorem 5.1, this was shown to contain a non-abelian free subgroup in [18]. However, one of the two corresponding groups in Table 2, namely Group 4a, is known to be finite of relatively small order [14].

3. If $W(x,y)$ is such that $W(uvu,v)$ satisfies the small-cancellation hypothesis of Corollary 2.3, then $W(x,y)$ satisfies the small-cancellation hypothesis of Corollary 2.2, but the converse does not hold in general.

4. If $W(x,y)$ has length parameter $k \in \{4,5,6\}$, then known results imply that the Rosenberger conjecture holds for G, as we saw in the proof of Theorem 5.1. But $W(uvu,v)$ has length parameter $2k \in \{8,10,12\}$ and existing results do not necessarily apply to H.

These remarks indicate that the $(2,3,2)$ situation, with even length, is somewhat more complicated than the $(3,3,2)$ case. We have not been able to prove the Rosenberger conjecture in its entirety for the $(2,3,2)$ case. Nevertheless, we have been able to reduce the number of potential counterexamples to 6.

Theorem 6.1 *Let $H = \langle u, v | u^2 = v^3 = Z(u,v)^2 = 1\rangle$ be a generalised triangle group, where $Z(u,v) = uv^{\gamma(1)} \cdots uv^{\gamma(2k)}$ and $\gamma(i) \in \{1,2\}$ for each i. Then the Rosenberger conjecture holds for H, except possibly when Z is, up to equivalence, one of the following:*

1. $(uv)^3(uv^2)^2uv(uv^2)^2uvuv^2$;

2. $(uv)^4(uv^2)^3(uv)^2uv^2$;

3. $(uv)^5(uv^2)^3(uv)^2uv^2uv(uv^2)^2$;

4. $(uv)^4(uv^2)^4uv(uv^2)^3(uv)^2uv^2uv(uv^2)^2$;

5. $(uv)^4(uv^2)^4uv(uv^2)^2uv(uv^2)^3(uv)^3(uv^2)^2uvuv^2$;

6. $(uv)^4(uv^2)^2uv(uv^2)^3(uv)^2uv^2uv(uv^2)^2$.

Proof. The proof of this theorem follows the same pattern as that of Theorem 5.1. The same computer search that produced Table 1 also produces a complete list (Table 2) of those words (up to equivalence) whose trace polynomials have the form indicated in Theorem 4.4. If W is not equivalent to a word in Table 2, then H contains a nonabelian free subgroup, by Theorem 4.4.

Table 2 is split into three parts. Part 3 contains the six exceptional words listed in the statement: we can prove nothing about the corresponding groups H.

Each word in part 2 of Table 2 satisfies the small-cancellation hypothesis of Corollary 2.3, so the corresponding group H contains a nonabelian free subgroup by Corollary 2.3.

Most of the groups in part 1 of table 2 can be handled by existing results. Specifically, groups 1a, 1b, 2, 3, 4a and 6 are known to be finite of the given orders [14, 17], while group 4b was shown to contain non-abelian free subgroups in [20] (by observing that its unique subgroup of index 2 is the group 4 in Table 1).

The remaining two groups can be dealt with by calculations using GAP [10] In group 5, the normal closure of $(uv)^{10}$ has index 7680 and is free abelian of rank 6, while in group 8 the normal closure of $(uv)^5$ has a non-abelian free homomorphic image of rank 3. (Logs of GAP sessions performing these calculations are shown in the Appendix.)

7 Computational Aspects

The main computational aspect of this work is the search for words with appropriate trace polynomials. By Lemma 4.2 the search in the $(3,3,2)$ case is essentially the same as that in the $(2,3,2)$ case with k even. In what follows we use the latter framework. Thus we put

$$G = \langle x, y | x^2 = y^3 = W(x,y)^2 = 1\rangle,$$

$$W(x,y) = xy^{\alpha(1)} \cdots xy^{\alpha(k)}$$

where k is even and $\alpha(j) \in \{1,2\}$ for each j.

We use the formulae in [15, Lemma 9] for the coefficients of $\tau_W(\lambda)$ to restrict the shape of the words for which we are searching. In our context, the coefficient of λ^{k-2} in $\tau_W(l)$, where $W = xy^{\alpha(1)} \cdots xy^{\alpha(k)}$, is $B_1 := b(1) + \cdots + b(k)$, where

$$b(j) := \begin{cases} -1 & \text{if } a(j) = a(j+1) \\ \frac{-1+i\sqrt{3}}{2} & \text{if } a(j) = 2 \neq a(j+1) \\ \frac{-1-i\sqrt{3}}{2} & \text{if } a(j) = 1 \neq a(j+1). \end{cases}$$

Moreover, the coefficient of l^{k-4} in $\tau_W(l)$ is $B_2 := \sum_{\{j,j'\}} b(j)b(j')$, where the sum is over all 2-element subsets $\{j,j'\}$ of $\{1,\ldots,k\}$ such that $j \neq j' \neq j \pm 1 \bmod k$.

Thus if we rewrite W in the form

$$W(x,y) = (xy)^{\_(1)}(xy^2)^{\gamma(1)} \cdots (xy)^{\_(m)}(xy^2)^{\gamma(m)}$$

with $\_(1) + \gamma(1) + \cdots + \_(m) + \gamma(m) = k$, then it follows from [15, Lemma 9] that $B_1 = m - k$, so

$$\tau_W(l) = l^k - (k-m)l^{k-2} + \cdots$$

In particular, if $\tau_W(l) = (l^2-1)^a(l^2-2)^b(l^4-3l^2+1)^c$ as required by Theorem 4.4, then $m = a+c$.

With the above calculation in mind, it is convenient to store the word W in the form of the list $L_W := [\_(1), \gamma(1), \ldots, \_(m), \gamma(m)]$ of positive integers. The GAP [10] command 'OrderedPartitions' produces all such lists.

At this point we encounter a software problem: the output of 'OrderedPartitions(40,20)', for example, should be a list of more than 2^{32} lists, which would exceed GAP's upper bound for list lengths. So the command 'OrderedPartitions(40,20)' will lead to an error. However, we can overcome this problem as follows.

It turns out from the formulae in [15, Lemma 9] that, just as the second coefficient B_1 of $\tau_W(l)$ determines the length of the list corresponding to W, the third coefficient B_2 determines the number of elements in that list which are equal to 1. To see this, note that

$$2B_2 = B_1^2 - \sum_{j=1}^{k} b(j)^2 - 2\sum_{j=1}^{k} b(j)b(j+1).$$

From k and B_1 we can calculate m, and hence determine how many of the $b(j)$ are equal to -1. This in turn enables us to calculate $\sum b(j)^2$. Hence, if we also know B_2, we can calculate $\sum b(j)b(j+1)$. On the other hand, the expression

$$W(x,y) = (xy)^{\_(1)}(xy^2)^{\gamma(1)} \cdots (xy)^{\_(m)}(xy^2)^{\gamma(m)}$$

enables us to count the number of j for which $b(j)b(j+1) \neq 1$: this happens precisely at the beginning and the end of each syllable $(xy)^{\_(i)}$ or $(xy^2)^{\gamma(i)}$ for which $\_(i) \geq 2$ (resp., $\gamma(i) \geq 2$). Reconciling this with our previous calculation of $\sum b(j)b(j+1)$ tells us how many entries in L_W are equal to 1.

We omit the details, but when

$$\tau_W(l) = (l^2-1)^a(l^2-2)^b(l^4-3l^2+1)^c$$

with $a, b \leq 1$ as in Theorem 4.4, the calculation shows that the resulting list L_W has

- length $2c+2$ with c entries equal to 1, if $a = b = 1$;
- length $2c+2$ with $c+1$ entries equal to 1, if $a = 1$ and $b = 0$;
- length $2c$ with $c-1$ entries equal to 1, if $a = 0$ and $b = 1$;
- length $2c$ with c entries equal to 1, if $a = b = 0$.

Now an ordered partition $\mathcal{P}$ of n into p positive integers, of which precisely q are equal to 1, can be completely described by two simpler partitions. The first is the partition of $n-p$ into $p-q$ parts, obtained by subtracting 1 from each element of $\mathcal{P}$ and then removing the zeroes. The second is a partition of p into $q+1$ positive integers, which encodes which entries of $\mathcal{P}$ are equal to 1. This gives us an algorithm implementable in GAP for conducting the search. Use 'OrderedPartitions' to create two lists of lists. For each pair in list1 $\times$ list 2, create a list that corresponds to a word. Calculate its trace polynomial: if this matches the form in Corollary 4.4 then add it to the output.

In practice we refine this algorithm in a number of ways.

1. We wish our output to contain only one word from each equivalence class. Ideally, we could select only one word from each equivalence class before calculating the trace polynomial, but this is not very efficient to do. A good compromise is to apply a fast-but-crude filter before the event (which will occassionally let through two or more equivalent words while ensuring that at least one word from each class gets through), and then to apply a less efficient but rigorous filter to the (much smaller) output data.

2. Before calculating trace polynomials, we apply a further filter to check that G admits the appropriate essential permutation representations (onto A_4, S_4, A_5). This reduces the number of matrix calculations that are required.

3. We replace the trace polynomial calculation by an amended version, that enables us to find the value of $\tau_W(l)$ for integer values of l using only matrices with integer entries (which is more efficient than doing matrix calculations over a polynomial ring). To test for correctness of the trace polynomial, it suffices to test its values at sufficiently many integer points.

Code implementing this algorithm is listed in the ancillary file *gtg232.g*.

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Appendix: GAP Sessions

1) The following GAP session considers Group 5 from Table 2

$$G_5 := \langle x, y | x^2 = y^3 = ((xy)^4(xy^2)^2xyxy^2)^2 = 1\rangle.$$

It demonstrates that G_5 has a free abelian normal subgroup N of rank 6 and index $7680 = 60 \cdot 2^7$.

A more detailed analysis shows that the kernel K of the essential representation $G_5 \to A_5$ has commutator subgroup $[K, K]$ of order 2, and $K/[K, K]$ is free abelian of rank 6. It follows that the subgroup K^2 generated by $\{g^2; g \in K\}$ is central in K and has index 2^7. The subgroup N above is precisely K^2.

```
gap> F:=FreeGroup(2);;
gap> x:=F.1;; y:=F.2;;
gap> W:=(x*y)^4 * (x*y^2)^2 * (x*y) * (x*y^2);;
```

```
gap> G:=F/[x^2,y^3,W^2];;
gap> Q:=F/[x^2,y^3,W^2,(x*y)^10];;
gap> Size(Q);
7680
gap> H:=Subgroup(G,[(G.1*G.2)^10]);;
gap> P:=PresentationNormalClosure(G,H);;
gap> SimplifyPresentation(P);
#I  there are 22 generators and 295 relators of total length 1240
#I  there are 9 generators and 77 relators of total length 380
#I  there are 6 generators and 27 relators of total length 160
#I  there are 6 generators and 22 relators of total length 114
gap> TzGoGo(P);
#I  there are 6 generators and 18 relators of total length 86
#I  there are 6 generators and 16 relators of total length 68
gap> TzPrint(P);
#I  generators: [ _x5, _x6, _x219, _x220, _x498, _x500 ]
#I  relators:
#I  1.  4  [ 3, 2, -3, -2 ]
#I  2.  4  [ -2, 5, 2, -5 ]
#I  3.  4  [ -6, -2, 6, 2 ]
#I  4.  4  [ -6, -4, 6, 4 ]
#I  5.  4  [ -4, -2, 4, 2 ]
#I  6.  4  [ -5, 3, 5, -3 ]
#I  7.  4  [ 3, 1, -3, -1 ]
#I  8.  4  [ 3, -6, -3, 6 ]
#I  9.  4  [ -3, 4, 3, -4 ]
#I  10.  4  [ 1, -6, -1, 6 ]
#I  11.  4  [ 4, 5, -4, -5 ]
#I  12.  4  [ 1, 5, -1, -5 ]
#I  13.  4  [ -5, -6, 5, 6 ]
#I  14.  4  [ 1, -2, -1, 2 ]
#I  15.  4  [ 4, -1, -4, 1 ]
#I  16.  8  [ -5, -2, 4, -3, 5, 2, -4, 3 ]
```

2) The following GAP session considers Group 8 from Table 2

$$G_8 := \langle x, y | x^2 = y^3 = ((xy)^4(xy^2)^3xyxy^2(xy)^2xy^2)^2 = 1\rangle.$$

It demonstrates that the kernel of the essential representation $G_8 \to A_5$ admits an epimorphism onto the free group of rank 3.

As an immediate consequence, it follows that the Group 8 from Table 1 also has a finite-index subgroup that admits an epimorphism onto a nonabelian free group.

```
gap> F:=FreeGroup(2);;
gap> x:=F.1;; y:=F.2;;
gap> W:=(x*y)^4 * (x*y^2)^3 * (x*y) * (x*y^2) * (x*y)^2 * (x*y^2);;
gap> G:=F/[x^2,y^3,W^2];;
gap> H:=Subgroup(G,[(G.1*G.2)^5]);;
```

```
gap> P:=PresentationNormalClosure(G,H);;
gap> gg:=GeneratorsOfPresentation(P);;
gap> for i in [1,3,7] do AddRelator(P,gg[i]); od;
gap> SimplifyPresentation(P);;
#I  there are 3 generators and 0 relators of total length 0
```

3) This GAP session shows that Group 7 in Table 1 is large, following the proof in [20].

$$G_7 = \langle x, y | x^3 = y^3 = (xyxyx^2y^2x^2yxy^2)^2 = 1\rangle.$$

It has a subgroup of index 12 which admits an epimorphism onto $\mathbb{Z} * \mathbb{Z}_2$.

```
gap> F:=FreeGroup(2);;
gap> x:=F.1;; y:=F.2;;
gap> W:=x*y*x*y*x^2*y^2*x^2*y*x*y^2;;
gap> G:=F/[x^3,y^3,W^2];;
gap> a:=G.1;; b:=G.2;;
gap> s1:=b*a^2;; s2:=a^2*b*a*b^2*a^2;;
gap> s3:=a*b*a*b^2*a^2*b*a;;
gap> s4:=a*b*a*b*a*b*a*b^2*a^2*b^2*a^2;;
gap> H:=Subgroup(G,[s1,s2,s3,s4]);;
gap> Index(G,H);
12
gap> P:=PresentationSubgroup(G,H);;
gap> gg:=GeneratorsOfPresentation(P);
[ _x1, _x2, _x3, _x4, _x5, _x6 ]
gap> AddRelator(P,gg[1]*gg[2]^-1*gg[1]*gg[2]^-1);
gap> AddRelator(P,gg[2]*gg[3]^-1);
gap> SimplifyPresentation(P);
#I  there are 2 generators and 1 relator of total length 4
gap> TzPrint(P);
#I  generators: [ _x1, _x2 ]
#I  relators:
#I  1.  4  [ -1, 2, -1, 2 ]
```

4) This GAP session uses the functions A5Poly, A4A5Poly, S4A5Poly, and A4S5A5Poly (see the ancillary file *gtg232.g* for function listings) to compute all words in $\langle x|x^2 = 1\rangle * \langle y|y^3 = 1\rangle$, up to equivalence, with trace polynomial of the form $(l^2-1)^a(l^2-2)^b(l^4-3l^2+1)^c$, where $a, b \leq 1$ and $c \leq 3(a+b+1)$. Words are output as even-length lists of positive integers: $[a(1), b(1), \ldots, a(t), b(t)]$ is shorthand for $(xy)^{a(1)}(xy^2)^{b(1)} \cdots (xy)^{a(t)}(xy^2)^{b(t)}$.

The time requirement of each of these GAP functions grows at least exponentially with the input parameter c. For small values of c, the runtime is essentially instantaneous. But the final run (corresponding to the case $a = b = 1, c = 9$) took close to 6 hours of CPU time on a 3GHz processor. Thus it appears that the theoretical limits supplied by Theorems 4.3 and 4.4 are not very far short of the practical limits for this impementation.

```
gap> A5Poly(1);
```

```
[ [ 3, 1 ] ]
gap> A5Poly(2);
[ [ 4, 1, 1, 2 ] ]
gap> A5Poly(3);
[ [ 4, 3, 1, 1, 2, 1 ] ]
gap> A4A5Poly(1);
[ [ 2, 1, 1, 2 ] ]
gap> A4A5Poly(2);
[ [ 3, 1, 1, 2, 1, 2 ] ]
gap> A4A5Poly(3);
[ [ 4, 2, 1, 1, 1, 2, 1, 2 ] ]
gap> A4A5Poly(4);
[ [ 4, 2, 1, 1, 2, 1, 1, 3, 1, 2 ] ]
gap> A4A5Poly(5);
[ [ 4, 2, 1, 2, 1, 3, 3, 1, 1, 2, 1, 1 ] ]
gap> A4A5Poly(6);
[  ]
gap> S4A5Poly(1);
[ [ 4, 2 ] ]
gap> S4A5Poly(2);
[ [ 4, 3, 2, 1 ] ]
gap> S4A5Poly(3);
[ [ 5, 3, 2, 1, 1, 2 ] ]
gap> S4A5Poly(4);
[ [ 4, 4, 2, 1, 1, 2, 3, 1 ] ]
gap> S4A5Poly(5);
[ [ 4, 4, 1, 1, 2, 3, 3, 1, 2, 1 ] ]
gap> S4A5Poly(6);
[  ]
gap> A4S4A5Poly(1);
[ [ 3, 2, 1, 2 ] ]
gap> A4S4A5Poly(2);
[  ]
gap> A4S4A5Poly(3);
[ [ 4, 2, 1, 1, 2, 3, 1, 2 ], [ 4, 3, 1, 2, 1, 1, 2, 2 ],
  [ 4, 3, 2, 2, 1, 2, 1, 1 ] ]
gap> A4S4A5Poly(4);
[ [ 4, 2, 1, 2, 3, 3, 1, 2, 1, 1 ], [ 4, 3, 1, 2, 1, 2, 3, 2, 1, 1 ] ]
gap> A4S4A5Poly(5);
[  ]
gap> A4S4A5Poly(6);
[ [ 4, 2, 1, 2, 1, 1, 3, 2, 1, 4, 3, 2, 1, 1 ],
  [ 4, 1, 1, 2, 3, 1, 1, 2, 1, 2, 3, 4, 1, 2 ],
  [ 4, 3, 1, 1, 2, 1, 2, 2, 3, 4, 1, 2, 1, 1 ],
  [ 4, 3, 1, 1, 2, 1, 3, 4, 1, 2, 1, 1, 2, 2 ] ]
gap> A4S4A5Poly(7);
[  ]
```

```
gap> A4S4A5Poly(8);
[  ]
gap> A4S4A5Poly(9);
[  ]
```

Table 1: Words in $\mathbb{Z}_3 * \mathbb{Z}_3$ with trace polynomial as in Theorem 4.3

	$W(x,y)$	$\tau(l)$	SCC
1	xy	l	NO
2	$xyxy^2$	l^2-l-1	NO
3	xyx^2y^2	$l(l-1)$	NO
4	$xyxyx^2y^2$	$l(l^2-l-1)$	NO
5	$xyxyx^2yx^2y^2$	$(l^2-l-1)^2$	NO
6	$xyxy^2x^2yx^2y^2$	$l(l-1)(l^2-l-1)$	NO
7	$xyxyx^2y^2x^2yxy^2$	$l(l^2-l-1)^2$	NO
8	$xyxyx^2y^2x^2yx^2yxy^2$	$(l^2-l-1)^3$	NO
9	$(xyxyx)(y^2x^2y^2x)(yx^2yx^2y^2)$	$l(l^2-l-1)^3$	YES
10	$(xyxy)(x^2y^2x^2yx)(y^2x^2yx^2y^2xy^2)$	$l(l-1)(l^2-l-1)^3$	YES
11	$(xyxy)(x^2y^2x^2yx^2)(y^2xy^2xyx^2y^2)$	$l(l-1)(l^2-l-1)^3$	YES
12	$(xyxy)(x^2y^2xy^2x^2y^2)(xyx^2yx^2y^2)$	$l(l-1)(l^2-l-1)^3$	YES
13	$(xyxy)(x^2y^2x^2y^2)(xy^2x^2y^2xyx^2yx^2y^2)$	$l(l^2-l-1)^4$	YES
14	$(xyxy)(x^2y^2xy^2x^2yxy)(x^2y^2x^2yx^2y^2xy^2)$	$l(l-1)(l^2-l-1)^4$	YES
15	$(xyxy)(x^2y^2x^2y^2)(xy^2x^2yx^2y^2x^2yxyx^2y^2xy^2)$	$l(l^2-l-1)^5$	YES
16	$(xyxyx^2y^2)(x^2yxy^2xy^2x^2y^2x^2)(yxy^2xyx^2yx^2y^2x^2yxy^2)$	$l(l-1)(l^2-l-1)^6$	YES
17	$(xyxyx^2y^2x^2)(yxy^2xyx^2yx^2y^2x^2yxy^2x)(y^2x^2y^2x^2yxy^2)$	$l(l-1)(l^2-l-1)^6$	YES
18	$(xyxy^2x^2yx)(yx^2y^2xy^2xyxy^2)(x^2y^2x^2yx^2yxy^2xyxy)$	$l(l-1)(l^2-l-1)^6$	YES
19	$(xyx^2y^2x^2yx^2)(y^2xy^2xyxy^2x^2)(y^2x^2yxy^2x^2yx^2yxy^2xy)$	$l(l-1)(l^2-l-1)^6$	YES

The final column indicates whether or not W satisfies the small-cancellation hypotheses of Corollary 2.2. In those cases where it does, the bracketing indicates a subdivision of W into three non-pieces of length ≥ 4: $W \equiv U_1 \cdot U_2 \cdot U_3$.

Table 2: Words in $\mathbb{Z}_2 * \mathbb{Z}_3$ with trace polynomial as in Theorem 4.4
(Numeration corresponds to related words in Table 1.)

Part 1: Short words. These groups are already known or can be easily analysed.

	$W(u,v)$	$\tau(l)$	Size of H
1a	$uvuv$	l^2-2	24
1b	$uvuv^2$	l^2-1	24
2	$uvuvuvuv^2$	l^4-3l^2+1	120
3	$uvuvuv^2uv^2$	$(l^2-1)(l^2-2)$	576
4a	$uvuvuvuvuv^2uv^2$	$(l^2-2)(l^4-3l^2+1)$	2880
4b	$uvuvuv^2uv^2uvuv^2$	$(l^2-1)(l^4-3l^2+1)$	large
5	$uvuvuvuvuv^2uv^2uvuv^2$	$(l^4-3l^2+1)^2$	abelian-by-finite
6	$uvuvuvuv^2uv^2uvuv^2uv^2$	$(l^2-1)(l^2-2)(l^4-3l^2+1)$	424673280
8	$uvuvuvuvuv^2uv^2uv^2uvuv^2uvuvuv^2$	$(l^4-3l^2+1)^3$	large

Part 2: Small cancellation words. Bracketing gives $W \equiv U_1 \cdot U_2 \cdot U_3$ as in Corollary 2.3.

	$W(u,v)$
9b	$(uvuvuvuv)(uv^2uv^2uvuv^2uv^2)(uvuv^2uvuv^2uv^2)$
10	$(uvuvuvuv)(uv^2uv^2uv^2uvuv)(uv^2uv^2uvuv^2uv^2uvuv^2)$
11	$(uvuvuvuv)(uv^2uv^2uv^2uvuv^2)(uv^2uvuv^2uvuvuv^2uv^2)$
13b	$(uvuvuvuv)(uv^2uv^2uvuv^2uv^2)(uv^2uvuv^2uvuvuv^2uvuv^2uv^2)$
14a	$(uvuvuvuv)(uv^2uv^2uvuv^2uv^2uvuv)(uvuv^2uv^2uv^2uvuv^2uv^2uvuv^2)$
14b	$(uvuvuvuv)(uv^2uv^2uv^2uvuv^2uv^2)(uvuv^2uv^2uvuvuvuv^2uv^2uvuv^2)$
15b	$(uvuvuvuv)(uv^2uv^2uvuv^2uv^2uv)(uv^2uv^2uv^2uvuvuvuv^2uvuv^2uv^2uvuv^2)$
16	$(uvuvuvuvuv^2uv^2)(uv^2uvuvuv^2uvuv^2uv^2uv^2uv^2)(uvuvuv^2uvuvuv^2uvuv^2uv^2uv^2uvuvuv^2)$
17	$(uvuvuvuvuv^2uv^2uv^2)(uvuvuv^2uvuvuv^2uvuv^2uv^2uv^2uvuvuv^2uv)(uv^2uv^2uv^2uv^2uvuvuv^2)$
18	$(uv^2uv^2uv^2uvuvuv^2uv^2)(uvuv^2uv^2uvuv^2uvuvuvuv^2)(uv^2uv^2uv^2uvuv^2uvuvuv^2uvuvuvuv)$
19	$(uvuv^2uvuvuv^2uvuvuvuv)(uv^2uv^2uv^2uvuv^2uv^2uvuv^2uvuvuv)(uv^2uv^2uv^2uv^2uvuvuv^2uv^2)$

Part 3. Cases remaining open

	$W(u,v)$
7a	$uvuvuvuvuv^2uv^2uv^2uvuvuv^2$
7b	$uvuvuvuv^2uv^2uvuv^2uv^2uvuv^2$
9a	$uvuvuvuvuvuv^2uv^2uv^2uvuvuv^2uvuv^2uv^2$
12	$uvuvuvuvuv^2vu^2uvuv^2uv^2uv^2uvuvuv^2uvuv^2uv^2$
13a	$uvuvuvuvuv^2uv^2uv^2uv^2uvu^2v^2uv^2uvuvuv^2uvuv^2uv^2$
15a	$uvuvuvuvuv^2uv^2uv^2uv^2uvuv^2uv^2uvuv^2uv^2uv^2uvuvuvuv^2uv^2uvuv^2$